Can one detect a non-smooth null infinity?

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Abstract

It is shown that the precession of a gyroscope can be used to elucidate the nature of the smoothness of the null infinity of an asymptotically flat spacetime (describing an isolated body). A model for which the effects of precession in the non-smooth null infinity case are of order $r^{-2} \ln r$ is proposed. By contrast, in the smooth version the effects are of order r^{-3}. This difference should provide an effective criterion to decide on the nature of the smoothness of null infinity.

1   Introduction

For sometime now, the smoothness or not of null infinity of an asymptotically flat spacetime has remained as an issue of debate in the mathematical studies of the Einstein field equations. *La raison d'être* of the asymptotically flat spacetimes is to provide a model to describe the gravitational field of isolated bodies within the framework of General Relativity. The way one could define an isolated body in GR has been the source of much, and in a way still present debate (see for example [6]). The most favoured solution was put forward by Penrose [11, 12], and consists of performing a conformal rescaling of the spacetime which results in the attachment of a boundary to the original manifold. Losely speaking, one will have a spacetime describing an isolated body, whenever such a "compactification procedure" can be performed, and the associated boundary (null infinity, $\mathscr{I}$ [1]) is a null hypersurface.

An outsider could regard the considerations on the smoothness of null infinity as just mere technical matters without real physical significance, beyond allowing the mathematical treatment of the problem. It is the objective of this article to point out that there are physical effects differentiating a spacetime with a smooth null infinity and one with a non-smooth one.

In order to make the discussion more precise, the forthcoming analysis will be centered in the class of asymptotically flat spacetimes with a non-smooth null infinity known as polyhomogeneous spacetimes. These spacetimes possess the peculiarity of having asymptotic expansions in terms of a parameter r and its logarithm, $\ln r$. The presence of logarithms in the asymptotic expansions yields as a result a non-smooth null infinity [4]. More recently, work on a new representation of spatial infinity has shown the appearance of logarithmic singularities at "the points where null infinity meets spatial infinity" [7] These logarithmic terms have a long history, and can also appear in similar expansion for linear fields[21]. As early as the late 50's independent work by Fock and Bonnor contained such logarithms. Some years later the seminal work by Sachs [14, 16] and Bondi et al. [1] invoked an "outgoing radiation condition" analogous to that of Sommerfeld (see [17] and also [5]) for the wave equation and the electromagnetic field in order to preclude the appearance of logarithms in the asymptotic expansions. There has been some confusion in the

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[1] The symbol $\mathscr{I}$ denoting null infinity is called "scri". Incidentally, the phonetic transliteration of scri into Polish is *skraj*, which curiously enough means boundary!

literature about the meaning and significance of these kind of radiation conditions. Sommerfeld's condition for the wave equation in Minkowski spacetime aims to exclude incoming radiation, and not to ensure the existence of outgoing radiation. Since the time this condition was formulated, it has been noticed that retarded solutions for spatially bounded sources satisfy Sommerfeld's condition automatically in future directions. However, these solutions do not necessarily satisfy these conditions in past null directions. If one wants to exclude incoming radiation, conditions have to be imposed precisely along the past null directions coming from past null infinity ($\mathscr{I}^-$). As one recedes towards past null infinity along null curves, one finds that the retarded field reflects the source behaviour at earlier times. Thus, a condition on the time dependence of the source in the infinite past is required so that the retarded field satisfies Sommerfeld's condition at $\mathscr{I}^-$ [22]. Conditions of this type have long time been given for the wave equation, the gravitational field and linearised gravity [10]. In the case of the full gravitational field, the situation is different because there are no integral representations of the field. The presence of incoming radiation at any time cannot be avoided (in a radiative spacetime, of course!), and can be interpreted as describing the phenomena of gravitational wave scattering and gravitational wave tails that die off suitably in a neighbourhood of null infinity.

Spacetimes with a smooth null infinity on the one hand, and polyhomogeneous spacetimes on the other, are two classes of solutions of the Einstein field equation which attempt to provide a description of the physics of isolated bodies. Now, given a particular physical system (which can be approximately considered as an isolated system), to which class should belong the solution describing it? A further related question is whether these classes are large enough to describe most systems of physical interest, or do we need to look for more general families of solutions? The answer to these questions must come through the comparison of the predictions given by each type of solution with the observations. In particular, one would like to have some specific physical effect which could be use as a clear "finger print"of a particular class of solutions.

One possible way of detecting the presence of gravitational radiation is through its effects on a gyroscope. Herrera & Hernández [9] have discussed how to extract information about gravitational waves by means of their effects on a gyroscope moving on a particular world line. Here, these ideas are taken over in order to show that in principle it is possible to make some statements on the nature of the null infinity of the spacetime. In order to do so, a simple (oscillatory) model for the news function of a radiative system is put forward. Under this assumption it is shown that the effects of precession of a gyroscope in a polyhomogeneous spacetime can be an order of magnitude stronger than those appearing in the peeling counterpart. This effect could be used to decide whether a radiative system should be modelled by means of a smooth or a non-smooth null infinity.

2   The Bondi metric

The early studies of the asymptotic behaviour of the gravitational radiation produced by an isolated body were carried out by means of the construction of an *ad hoc* metric and associated coordinate system [1]. The referred metric is now widely known as the axially symmetric Bondi metric. In the most standard parametrisation it reads as follows:

$$\mathrm{d}s^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right)\mathrm{d}u^2 + 2e^{2\beta}\mathrm{d}u\mathrm{d}r + 2Ur^2 e^{2\gamma}\mathrm{d}u\mathrm{d}\theta - r^2\left(e^{2\gamma}\mathrm{d}\theta^2 + e^{-2\gamma}\sin^2\theta\mathrm{d}\varphi^2\right). \quad (1)$$

The coordinate u is a retarded time labelling (future oriented) light cones, and (θ, φ) are the usual angular coordinates of the celestial sphere. The crucial ingredient of the Bondi metric is the coordinate r, usually known as luminosity parameter. It satisfies

$$r^4 \sin^2 \theta = \det h_{ij}, \quad (2)$$

where h_{ij} is the angular part of the metric. This metric assumes the existence of an hypersurface orthogonal axial Killing vector field. In addition, the metric functions are required to satisfy some regularity conditions on the poles. But these, shall not concern us here. The generalisation of eqn.(1) to the case where no symmetries are present was studied by Sachs [15].

The present analysis will be restricted to the axial symmetric case, mainly in order to ease the calculations. Nevertheless, as pointed out by Sachs himself [15], no interesting physics lost with this assumption as all the relevant features of the gravitational radiation phenomena are generally already present in the axially symmetric case.

3   Polyhomogeneous spacetimes

The asymptotic expansions of the metric coefficients of the Bondi metric for the case of a peeling asymptotically flat spacetime are very well known. Their leading terms read:

$$\gamma_* = c_0 r^{-1} + O(r^{-3}), \tag{3}$$

$$\beta_* = -\frac{1}{4}c_0^2 r^{-2} + O(r^{-4}), \tag{4}$$

$$U_* = -(c_{0\theta} + 2c_0 \cot\theta)r^{-2} + O(r^{-3}), \tag{5}$$

$$V_* = r - 2M + O(r^{-1}). \tag{6}$$

Note here the absence of the r^2-term in the expansion of γ. This is the way the "outgoing radiation condition" (cfr. the introduction) is imposed.

The most natural and simple example of a polyhomogeneous spacetime is the so-called minimally polyhomogeneous for which [2]:

$$\gamma_\# = cr^{-1} + \gamma_2 r^{-2} + O(r^{-3}\ln r). \tag{7}$$

The presence of the extra term γ_2 in the expansion gives rise to logarithmic terms at higher order $O(r^{-3})$ in γ and at order $O(r^{-4})$, $O(r^{-3})$, and $O(r^{-1})$ in the metric functions β, U, and V respectively. Looking at these spacetimes in the Newman-Penrose framework one sees that $\Psi_0 = O(r^{-4})$, i.e. these spacetimes are non-peeling [20]. In the present article, a more general type of polyhomogeneous spacetime will be used. Namely that one generated by a metric function γ of the form:

$$\gamma_\# = (c_1 \ln r + c_2)\, r^{-1} + (\gamma_{21} \ln r + \gamma_{20})\, r^{-2} + O(r^{-3}\ln r). \tag{8}$$

In this case, the leading terms of the remaining metric functions are found to be:

$$\beta_\# = \frac{1}{4}\left(-c_1^2 \ln^2 r + \left(c_1^2 - 2c_1 c_0\right)\ln r - c_0^2 + c_0^2 + c_1 c_0 - \frac{1}{2}c_1^2\right)r^{-2} + O(r^{-3}\ln^2 r), \tag{9}$$

$$U_\# = \left((-c_{1\theta} - 2\cot\theta c_1)\ln r - c_{0\theta} + \frac{3}{2}c_{1\theta} - 2\cot\theta c_0 + 3\cot\theta c_1\right)r^{-2} + O(r^{-3}\ln r), \tag{10}$$

$$V_\# = r - 2M + (c_{1\theta\theta} + 3\cot\theta c_{1\theta} - 2c_1) + O(r^{-1}\ln r). \tag{11}$$

It can be shown that the coefficient c_1 is a constant of motion, i.e. $\partial_u c_1 = 0$ [4, 19]. This more general class of spacetimes is such that $\Psi_0 = O(r^{-3})$, and remarkably does possess a well defined Bondi mass [2, 3].

4   A model of a gyroscope

Consider an observer with world line given by $r = const$, $\theta = const$, $\varphi = const$. This observer has a 4-velocity given by:

$$u_a = \left(A, e^{2\beta}A^{-1}, Ur^2 e^{2\gamma}A^{-1}, 0\right), \tag{12}$$

with

$$A = \sqrt{Ve^{2\beta}r^{-1} - U^2 r^2 e^{2\gamma}}. \tag{13}$$

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[2]The subindex $_\#$ will be used to denote functions with polyhomogeneous expansions, while $_*$ will be used to denote functions with analytic expansions in $1/r$.

The precession of a gyroscope moving along such a world line is quantified through the vorticity of a congruence of world lines surrounding our fiduciary world line (for a discussion on the description of precession effects we refer to [13], and more recently to [8]). The vorticity vector is given by:

$$\omega^a = \frac{1}{2\sqrt{-g}}\varepsilon^{abcd}u_b u_{c,d}. \tag{14}$$

Thus, for the Bondi metric one has that [9]:

$$
\begin{aligned}
\Omega &= \sqrt{-\omega_a \omega^a} \\
&= \frac{1}{2}r^{-1}e^{-2\beta-\gamma}\left[2\beta_\theta e^{2\beta} - 2e^{2\beta}A_\theta A^{-1} - (Ur^2 e^{2\gamma})_r + 2Ur^2 e^{2\gamma}A_r A^{-1}\right. \\
&\quad \left. + e^{2\beta}(Ur^2 e^{2\gamma})_u - 2\beta_u e^{2\beta+2\gamma}Ur^2 A^{-2}\right].
\end{aligned}
\tag{15}
$$

The scalar Ω measures the rate of rotation with respect to the proper time of world lines of points with $r = const.$, $\theta = const.$, $\varphi = const.$. Thus, it describes the precession of a gyroscope moving along the fiduciary world line.

For a peeling spacetime one has that Ω is given by,

$$\Omega_* = -\frac{1}{2r}\left(\partial_u\partial_\theta c + \cot\theta\partial_u c\right) + O(r^{-2}). \tag{16}$$

5   A model to study precession

The effect of a gravitational wave on a ring of test particles in a plane perpendicular to the direction of propagation of the wave is well known. It can be deduced from the geodesic deviation equation. In this case it reads:

$$\delta\ddot{x}^a = -\frac{1}{2}\ddot{c}_0\left(e_2^a e_{2c} - e_1^a e_{1c}\right)\delta x^c \tag{17}$$

where e_1^a is an azimuthal vector, and e_2^a is a polar one. Due to the axial symmetry, the North Pole and the South Pole of the Celestial sphere are already fixed. Furthermore, due to the orthogonal transitivity of the axial Killing vector, there is only (+)-polarisation (the axes of stretching and contraction lie along the e_1^a and e_2^a directions).

In order to study the possible effects of the gravitational radiation on a gyroscope, the following model for the expansion coefficient c_0 is put forward:

$$c_0 = \frac{k}{\omega^2}\sin(\omega u), \tag{18}$$

where k is a constant, and ω is an arbitrary period. Thus, this model produces periodic oscillatory distortions of our ring of test particles. Its derivative yields the news function. This particular form of the news function will be used for both the peeling and polyhomogeneous cases. It is very important to point that this kind of periodic behaviour is an approximation that can only be valid for a limited period of retarded time. The radiative system should settle to a quiescent stationary state as one approaches future time infinity. In other words, there are no exactly periodic radiative spacetimes.

For concreteness, we will only be interested in analyzing what happens at the equatorial plane, i.e. $\theta = \pi/2$. Thus, it will not worry us that the function c_0 here suggested does not satisfy the regularity requirements at the poles.

Substitution into 15 and evaluating at the equatorial plane one gets:

$$\Omega_* = \partial_\theta M r^{-2} + O(r^{-3}), \tag{19}$$

that is, there are no $1/r$ precession effects. Those at order $1/r^2$ depend only the "polar asymmetry" of the mass aspect of the spacetime. The mass aspect, M, is related to the news function via:

$$M_u = -c_{0u} + \frac{1}{2}\left(c_{0\theta\theta} + 3c_{0\theta}\cot\theta - 2c_0\right)_u, \tag{20}$$

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This equation holds for both the peeling and the polyhomogeneous spacetimes. Thus one has that

$$M = -\frac{uk^2}{2\omega^2} - \frac{k}{\omega}\sin(\omega u) - \frac{k^2}{\omega^3}\sin(2\omega u) + f(\theta), \qquad (21)$$

where $f(\theta)$ is an arbitrary function coming out of the integration. If one particularises the model further more by requiring $\partial_\theta M = 0$, i.e. $f(\theta)$, then one arrives to a situation where there are no $1/r^2$ effects!

Now, in the case of a polyhomogeneous $\mathscr{I}$ one gets,

$$\Omega_\# = \left(\frac{5}{3}\partial_\theta c_1 - \frac{1}{3}\partial_{\theta\theta}c_1 + \frac{1}{6}\partial_u c_0 \partial_\theta c_1 + \partial_{\theta\theta}\right)r^{-2}\ln r + O(r^{-2}). \qquad (22)$$

Hence, there is an $r^{-2}\ln r$ order precession effect on a gyroscope moving along the fiduciary world line, while the effects in a peeling spacetime appear at order r^{-2} (or r^{-3} if the mass aspect happens to be isotropic!). This provides an effective criterion to differentiate the two classes of null infinity.

6   Conclusions

It has been shown that the precession of a gyroscope can be used to investigate the nature of the null infinity modelling an isolated gravitational system. The analysed model is quite simple, but nevertheless it seems to shed some light on this issue. And therefore providing some physical interpretation to some issues that may seem of a purely mathematical nature.

There has been some discussion on the nature of the physical interpretation of the logarithmic terms appearing in the expansions of asymptotically flat spacetimes. In [18] these terms have been regarded as associated to incoming radiation and wave tails. Herrera & Hernández [9] have argued that the $1/r^2$ effects are somehow related to incoming radiation in the form of wave tails. These ideas seem to bring further support to the relation between incoming radiation and logarithmic terms in asymptotic expansions.

Acknowledgements

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